# Monge-Ampère Structures and the Geometry of Incompressible Flows

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B. Banos[*], V. N. Roubtsov[†], & I. Roulstone[‡]

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**Abstract**

We show how a symmetry reduction of the equations for incompressible hydrodynamics in three dimensions leads naturally to Monge-Ampère structure, and Burgers'-type vortices are a canonical class of solutions associated with this structure. The mapping of such solutions, which are characterised by a linear dependence of the third component of the velocity on the coordinate defining the axis of rotation, to solutions of the incompressible equations in two dimensions is also shown to be an example of a symmetry reduction. The Monge-Ampère structure for incompressible flow in two dimensions is shown to be hyper-symplectic.

## 1 Introduction

The incompressible Euler and Navier–Stokes equations pose an enduring challenge for mathematical analysis. Previous works by Roubtsov and Roulstone (1997, 2001), Delahaies and Roulstone (2010), and by McIntyre and Roulstone (2002), have shown how quite sophisticated ideas from modern differential geometry offer a framework for understanding how coherent structures in large-scale atmosphere-ocean flows, such as cyclones and fronts, may be modelled using a hierarchy of approximations to the Navier–Stokes equations. These studies focussed on semi-geostrophic (SG) theory

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and related models. A Monge-Ampère (MA) equation lies at the heart of these models, and it was observed that geometry is the natural language for describing the interconnections between the transformation theory of the MA equation (Kushner *et al.* 2007), the Hamiltonian properties of fluid dynamical models, and particular features of solutions such as stability and discontinuity.

In this paper, we extend the more recent work of Roulstone *et al.* (2009a,b) on the Euler and Navier–Stokes equations, and show how the geometric approach to SG theory may be applied to the incompressible fluid equations in three dimensions: in particular, we show how vortex flows of Burgers' type are described by a Monge-Ampère structure. We also show how the partial differential equations that describe such canonical flows are obtained via a symmetry reduction.

This paper is organised as follows. We start by considering Monge-Ampère structures in four dimensions and incompressible flow in two dimensions. In Section 3, we show how a hyper-symplectic structure is associated with these equations. In Section 4, we review Monge-Ampère structures in six dimensions and the reduction principle. In Section 5 we show how Burger's vortices in three-dimensional flow relate to Monge-Ampère structures in six dimensions. We then explain how a symmetry reduction maps this to a Monge-Ampère, hyper-symplectic, structure in four dimensions. This reduction is surely known to fluid dynamicists in the context of the Lundgren transformation (Lundgren 1982). In Section 6 we show how a symmetry reduction of the incompressible Euler equations in three dimensions yields the generic geometric framework exploited in Section 5. A summary is given in Section 7.

# 2 The geometry of Monge-Ampère equations

## 2.1 Monge-Ampère structures

Lychagin (1979) proposed a geometric approach to to study of MA equations using differential forms. The idea is to associate with a form $\omega \in \Lambda^n(T^*\mathbb{R}^n)$, where $\Lambda^n$ denotes the space of differential $n$-forms on $T^*\mathbb{R}^n$, the MA equation $\Delta_\omega = 0$, where $\Delta_\omega : C^\infty(\mathbb{R}^n) \to \Omega^n(\mathbb{R}^n) \cong C^\infty(\mathbb{R}^n)$ is the differential operator defined by

$$\Delta_\omega(\phi) = (d\phi)^*\omega\,,$$

and $(d\phi)^*\omega$ denotes the restriction of $\omega$ to the graph of $d\phi$ ($d\phi : \mathbb{R}^n \to T^*\mathbb{R}^n$ is the differential of $\phi$).

Following Lychagin *et al.* (1993), a MA equation in $n$ variables can be described by a pair $(\Omega, \omega) \in \Lambda^2(T^*\mathbb{R}^n) \times \Lambda^n(T^*\mathbb{R}^n)$ such that

i) $\Omega$ is symplectic that is nondegenerate and closed

ii) $\omega$ is effective, that is $\omega \wedge \Omega = 0$.

Such a pair is called a Monge-Ampère structure.

A generalized solution of a MA equation is a lagrangian submanifold $L$, on which $\omega$ vanishes; that is, $L$ is an $n$-dimensional submanifold such that $\Omega|_L = 0$ and $\omega|_L = 0$. Note that if $\phi(x)$ is a regular solution then its graph $L_\phi = \{(x, \phi_x)\}$ in the phase space is a generalized solution.

In four dimensions (that is $n = 2$), a geometry defined by this structure can be either complex or real and this distinction coincides with the usual distinction between elliptic and hyperbolic differential equations in two variables. Indeed, when $\omega \in \Lambda^2(T^*\mathbb{R}^2)$ is a nondegenerate 2-form ($\omega \wedge \omega \neq 0$), one can associate with the Monge-Ampère structure $(\Omega, \omega) \in \Lambda^2(T^*\mathbb{R}^2) \times \Lambda^2(T^*\mathbb{R}^2)$ the tensor $I_\omega$ defined by

$$\frac{1}{\sqrt{|\operatorname{pf}(\omega)|}}\omega(\cdot, \cdot) = \Omega(I_\omega \cdot, \cdot) \tag{1}$$

where $\operatorname{pf}(\omega)$ is the pfaffian of $\omega$: $\omega \wedge \omega = \operatorname{pf}(\omega)(\Omega \wedge \Omega)$. This tensor is either an almost complex structure or an almost product structure:

a) $\Delta_\omega =$ is elliptic $\Leftrightarrow \operatorname{pf}(\omega) > 0 \Leftrightarrow I_\omega^2 = -Id$

b) $\Delta_\omega =$ is hyperbolic $\Leftrightarrow \operatorname{pf}(\omega) < 0 \Leftrightarrow I_\omega^2 = Id$

Lychagin and Rubtsov (1983a,b, 1993) (see also Kushner *et al.* (2007) for a comprehensive account of this theory) have explained the link there is between the problem of local equivalence of MA equations in two variables and the integrability problem of this tensor $I_\omega$:

**PROPOSITION 1.** *(Lychagin-Roubtsov theorem) The three following assertions are equivalent:*

1. $\Delta_\omega = 0$ *is locally equivalent to one of the two equations*

$$\begin{cases} \Delta\phi = 0 \\ \Box\phi = 0 \end{cases}$$

2. *the almost complex (or product) structure* $I_\omega$ *is integrable*

3. *the form* $\frac{\omega}{\sqrt{|\operatorname{pf}(\omega)|}}$ *is closed.*

## 3 The geometry of the incompressible Euler equations in two dimensions

### 3.1 The Poisson equation for the pressure

The incompressible Euler equations (in two or three dimensions) are

$$\frac{\partial u_i}{\partial t} + u_j u_{ij} + p_i = 0, \quad u_{ii} = 0, \tag{2}$$

with $p_i = \partial p/\partial x_i$, $u_{ij} = \partial u_i/\partial x_j$, and the summation convention is used. By applying the incompressibility condition, we find that the Laplacian of the pressure and the velocity gradients are related by

$$\Delta p = -u_{ij}u_{ji}. \tag{3}$$

In two dimensions, the incompressibility condition implies

$$u_1 = -\frac{\partial \psi}{\partial x_2} = -\xi_2, \;\; u_2 = \frac{\partial \psi}{\partial x_1} = \xi_1,$$

where $\psi(x_1, x_2, t)$ is the stream function and (3) becomes

$$\psi_{x_1x_1}\psi_{x_2x_2} - \psi^2_{x_1x_2} = \frac{\Delta p}{2}. \tag{4}$$

Equation (3) is normally used as a Poisson equation for the pressure. However, formally, we may consider (4) to be an equation of Monge-Ampère type for the stream function, if the Laplacian of the pressure is given. Roulstone *et al.* (2009a) discuss the relationship between the sign of the Laplacian of the pressure, the elliptic/hyperbolic type of the associated MA equation, and the balance between rate of strain and the enstrophy of the flow. This is related to the Weiss criterion (Weiss 1991), as discussed by Larcheveque (1993).

### 3.2 Hypersymplectic geometry

Following Section 2.1, (4) can be associated with a Monge-Ampère structure $(\Omega, \omega)$ on the phase space $M = T^*\mathbb{R}^2$, where $\Omega = dx_1 \wedge du_2 + du_1 \wedge dx_2$ is a symplectic form (which is written conventionally in the form $\Omega = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2$) and $\omega \in \Omega^2(M)$ is the 2-form defined by

$$\omega = du_1 \wedge du_2 - a(x_1, x_2)dx_1 \wedge dx_2$$

with

$$a(x_1, x_2, t) = \frac{\Delta p}{2}.$$

(Note that time, $t$, is a parameter in our examples, and its presence will not be indicated, unless crucial.)

We observe that

$$d\omega = 0 \tag{5}$$

and

$$\omega \wedge \omega = a\,\Omega \wedge \Omega. \tag{6}$$

Moreover, the tensor $A_\omega$ (cf. (1)) defined by $\omega(\cdot,\cdot) = \Omega(A_\omega\cdot,\cdot)$ satisfies

$$A_\omega^2 = -a.$$

Following Roubtsov and Roulstone (2001) (see also Kossowski (1992)), we define a symmetric tensor $g_\omega$ on $M$

$$g_\omega(X,Y) = \frac{2\,(\iota_X\omega \wedge \iota_Y\Omega + \iota_Y\omega \wedge \iota_X\Omega) \wedge dx_1 \wedge dx_2}{\Omega^2}.$$

In coordinates, one has

$$g_\omega = dx_1 \otimes du_2 - dx_2 \otimes du_1 \tag{7}$$

and therefore the signature of $g_\omega$ is $(2,2)$. Using this split metric we define a dual form $\hat{\omega} \in \Lambda^2(M)$

$$\hat{\omega}(\cdot,\cdot) = g_\omega(A_\omega\cdot,\cdot),$$

which in coordinates is

$$\hat{\omega} = -du_1 \wedge du_2 - a(x_1,x_2)dx_1 \wedge dx_2. \tag{8}$$

This dual form satisfies

$$d\hat{\omega} = 0 \tag{9}$$

and

$$\hat{\omega} \wedge \hat{\omega} = -a\,\Omega \wedge \Omega. \tag{10}$$

Hence, introducing the normalized form $\tilde{\Omega} = \sqrt{|a|}\Omega$, we obtain a triple $\left(\tilde{\Omega}, \omega, \hat{\omega}\right)$, which is hypersymplectic (Hitchin 1990) when $\Delta p \neq 0$:

$$\omega^2 = -\hat{\omega}^2, \quad \omega^2 = \varepsilon\tilde{\Omega}^2, \quad \hat{\omega}^2 = -\varepsilon\tilde{\Omega}^2,$$
$$\omega \wedge \hat{\omega} = 0, \quad \omega \wedge \tilde{\Omega} = 0, \quad \hat{\omega} \wedge \tilde{\Omega} = 0,$$

with $\varepsilon = 1$ if $\Delta p > 0$ and $\varepsilon = -1$ if $\Delta p < 0$.

Equivalently, we obtain three tensors $S$, $I$ and $T$ defined by

$$\hat{\omega}(\cdot,\cdot) = \omega(S\cdot,\cdot), \quad \omega(\cdot,\cdot) = \tilde{\Omega}(I\cdot,\cdot), \quad \hat{\omega}(\cdot,\cdot) = \tilde{\Omega}(T\cdot,\cdot),$$

which satisfy

$$\begin{aligned} &S^2 = 1, I^2 = -\varepsilon, T^2 = \varepsilon, \\ &TI = -IT = S, \\ &TS = -ST = I, \\ &IS = -SI = T. \end{aligned}$$

This structure should be compared with the hyper-Kähler properties of the SG equations (Delahaies and Roulstone 2010), and it extends the results presented in Roulstone *et al.* (2009a).

Moreover, since

$$d\omega = d\hat{\omega} = 0 \ \text{ and } \ d\tilde{\Omega} = \frac{1}{2\sqrt{a}}\, da \wedge \Omega,$$

we deduce from Lychagin-Roubtsov theorem (see Proposition 1) that

i) the product structure $S$ is always integrable

ii) the complex structure $I$ and the product structure $T$ (or vice versa) are integrable if and only if $\Delta p$ is constant.

### 3.3 Generalized solutions

A generalized solution of (4) is a 2d-submanifold $L^2 \subset M^4$ which is bilagrangian with respect to $\omega$ and $\Omega$. Since $\omega = \Omega(I,\cdot,\cdot)$, it is equivalent to saying that $L$ is closed under $I$: for any non vanishing vector field $X$ on $L$, $\{X, IX\}$ is a local frame of $L$.

**PROPOSITION 2.** *Let $L$ be a generalized solution and $h_\omega$ the restriction of $g_\omega$ on $L$.*

1. *if $\Delta p > 0$, the metric $h_\omega$ has signature $(2,0)$ or $(0,2)$*
2. *if $\Delta p < 0$, the metric $h_\omega$ has signature $(1,1)$.*

*Proof.* Let $X$ a local non vanishing vector field, tangent to $L$. Then $\{X, IX\}$ is a local frame of $L^2$ and $\{X, IX, SX, TX\}$ a local frame of $M^4$. Let $\alpha = \hat{\omega}(X, IX)$. Then we have

i) $\omega(X, X) = 0$ by skew-symmetry .

ii) $\omega(X, IX) = 0$ since $X$ and $IX$ are tangent to $L$.

iii) $\omega(X, SX) = -\omega(SX, X) = \hat{\omega}(X, X) = 0$.

iv) $\omega(X, TX) = \omega(SIX, X) = \hat{\omega}(IX, X) = -\alpha$.

Therefore, since $\omega$ is non degenerate, $\alpha$ cannot be 0. Moreover,

i) $g_\omega(IX, IX) = \dfrac{1}{|a|} g_\omega(A_\omega X, A_\omega X) = \dfrac{1}{|a|}\hat{\omega}(X, A_\omega X) = \dfrac{\alpha}{\sqrt{|a|}}$

ii) $g_\omega(IX, X) = \dfrac{1}{\sqrt{|a|}} g_\omega(A_\omega X, X) = \dfrac{1}{\sqrt{|a|}}\hat{\omega}(X, X) = 0$

iii) $g_\omega(X, X) = -\dfrac{1}{a} g_\omega(A_\omega^2 X, X) = -\varepsilon \dfrac{1}{|a|}\hat{\omega}(A_\omega X, X) = \dfrac{\alpha\varepsilon}{\sqrt{|a|}}$

We therefore obtain the matrix of $h_\omega$ in the representation $\{X, IX\}$:

$$h_\omega = \frac{\alpha}{\sqrt{|a|}} \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$$

$\square$

**REMARK 1.** *If $L = L_\psi$ is a regular solution, then the induced metric $h_\omega$ is affine. Indeed its tangent space is generated by $X_1 = \partial_{x_1} - \psi_{x_1 x_2}\partial_{u_1} + \psi_{x_1 x_1}\partial_{u_2}$ and $X_2 = \partial_{x_2} - \psi_{x_2 x_2}\partial_{u_1} + \psi_{x_1 x_2}\partial_{u_2}$. The induced metric on $L_\psi$ is therefore*

$$h_\omega = 2 \begin{pmatrix} \psi_{xx} & \psi_{xy} \\ \psi_{xy} & \psi_{yy} \end{pmatrix} \tag{11}$$

*and consequently the invariants of this tensor are*

$$\det(h_\omega) = 2\Delta p, \quad \operatorname{tr}(h_\omega) = 2\Delta\psi.$$

We note that metric, as written in (11), is related to the velocity gradient tensor (VGT) of a two-dimensional flow. The signature of the induced metric changes when the sign of the Laplacian of the pressure changes, and this is related to the flip between the elliptic/hyperbolic nature of (4) when viewed as a MA equation for $\psi$.

The VGT is not a symmetric tensor and its trace is the divergence of the flow (which equals zero in the incompressible case). The trace of $h_\omega$ is proportional to the vorticity of the flow. The determinant of the VGT is the same as $\det(h_\omega)$, and this is precisely the balance between the rate-of-strain and the enstrophy (vorticity squared). It is possible to construct equations of motion for the invariants of the VGT and for the invariants of $h_\omega$: see Roulstone *et al.* (2014) for further details.

# 4 Monge-Ampère structures in six dimensions

## 4.1 Geometry of 3-forms

In six dimensions ($n = 3$), there is again a correspondence between real/complex geometry and "nondegenerate" Monge-Ampère structures.

Lychagin *et al.* (1993) associated with a Monge-Ampère structure $(\Omega, \omega)$ an invariant symmetric form

$$g_\omega(X, Y) = \frac{\iota_X \omega \wedge \iota_Y \omega \wedge \Omega}{\mathrm{vol}}$$

whose signature distinguishes the different orbits of the symplectic group action.

In a seminal paper on the geometry of 3-forms, Hitchin (2001) defined the notion of nondegenerate 3-forms on a 6-dimensional space and constructed a scalar invariant, which we call the *Hitchin pfaffian*, which is non zero for such nondegenerate 3-forms. Hitchin also defined an invariant tensor $A_\omega$ on the phase space satisfying

$$A_\omega^2 = \lambda(\omega)\, Id.$$

Note that in the nondegenerate case, $K_\omega = \dfrac{A_\omega}{\sqrt{|\lambda(\omega)|}}$ is a product structure if $\lambda(\omega) > 0$ and a complex structure if $\lambda(\omega) < 0$.

Following Hitchin *op. cit.*, we observe

1. $\lambda(\omega) > 0$ if and only if $\omega$ is the sum of two decomposable forms; i.e.

$$\omega = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \beta_1 \wedge \beta_2 \wedge \beta_3$$

2. $\lambda(\omega) < 0$ if and only if $\omega$ is the sum of two decomposable complex forms; i.e.

$$\omega = (\alpha_1 + i\beta_1) \wedge (\alpha_2 + i\beta_2) \wedge (\alpha_3 + i\beta_3) + (\alpha_1 - i\beta_1) \wedge (\alpha_2 - i\beta_2) \wedge (\alpha_3 - i\beta_3)$$

Banos (2002) noted that on a 6-dimensional symplectic space, nondegenerate 3-forms correspond to nondegenerate Lychagin-Roubtsov symmetric tensors. Moreover the signature of $g_\omega$ is $(3,3)$ if $\lambda(\omega)>0$ and $(6,0)$ or $(4,2)$ if $\lambda(\omega)<0$.

Banos *op. cit.* established the following simple relation between the Hitchin and the Lychagin-Roubsov invariants

$$g_\omega(A_\omega\cdot,\cdot)=\Omega(\cdot,\cdot).$$

There is a correspondence between non-degenerate MA structures $(\Omega,\omega)$ and a "real Calabi-Yau structure" if the Hitchin pfaffian $\lambda(\omega)$ is positive, and a "complex Calabi-Yau structure" if the Hitchin pfaffian is negative.

For example, the Monge-Ampère structure associated with the "real" MA equation in three variables $(x_1,x_2,x_3)$

$$\mathrm{hess}(\phi)=1\,,$$

is the pair

$$\begin{cases}\Omega=dx_1\wedge d\xi_1+dx_2\wedge d\xi_2+dx_3\wedge d\xi_3\\ \omega=d\xi_1\wedge d\xi_2\wedge d\xi_3-dx_1\wedge dx_2\wedge dx_3\end{cases}$$

and the underlying real analogue of Kähler structure on $T^*\mathbb{R}^3$ in the coordinates $(x_1,x_2,x_3,\xi_1,\xi_2,\xi_3)$ is

$$\begin{cases}g_\omega=\begin{pmatrix}0&Id\\Id&0\end{pmatrix},\\[2ex] K_\omega=\begin{pmatrix}Id&0\\0&-Id\end{pmatrix}.\end{cases}$$

The Monge-Ampère structure associated with the special lagrangian equation

$$\Delta\phi-\mathrm{hess}(\phi)=0$$

is the pair

$$\begin{cases}\Omega=dx_1\wedge d\xi_1+dx_2\wedge d\xi_2+dx_3\wedge d\xi_3\\ \omega=\mathrm{Im}\big((dx_1+id\xi_1)\wedge(dx_2+id\xi_2)\wedge(dx_3+id\xi_3)\big)\end{cases}$$

and the underlying Kähler structure is the canonical Kähler structure on $T^*\mathbb{R}^3=\mathbb{C}^3$:

$$\begin{cases}g_\omega=\begin{pmatrix}Id&0\\0&Id\end{pmatrix},\\[2ex] K_\omega=\begin{pmatrix}0&-Id\\Id&0\end{pmatrix}.\end{cases}$$

As in the 4-dimensional case, there is a clear connection between the problem of local equivalence of MA equation in three variables and the integrability problem of generalized Calabi-Yau structures on $\mathbb{R}^6$ (Banos 2002):

**PROPOSITION 3.** *The three following assertions are equivalent:*

*i) the MA equation $\Delta_\omega = 0$ is locally equivalent to one of the three equations*

$$\begin{cases} \mathrm{hess}(\phi) = 1, \\ \Delta\phi - \mathrm{hess}(\phi) = 0, \\ \Box\phi + \mathrm{hess}(\phi) = 0, \end{cases}$$

*ii) the underlying generalized Calabi-Yau structure on $T^*\mathbb{R}^3$ is integrable,*

*iii) the form $\frac{\omega}{\sqrt[4]{|\lambda(\omega)|}}$ and its dual form (in the sense of Hitchin) are closed. Moreover, the metric $g_\omega$ should be flat.*

The generalized Calabi-Yau structure discussed here is defined in Banos (2002). This notion is different from the generalized Calabi-Yau structures in sense of N. Hitchin and M. Gualtieri. These relations are clarified and discussed further in §14.4 of Kosmann-Schwarzbach and Rubtsov (2010).

It is important to note that the geometry associated with a MA equation $\Delta_\omega = 0$ of real type ($\lambda(\omega) > 0$) is essentially real but it is very similar to the classic Kähler geometry. In particular, when this geometry is integrable, there exists a potential $\Phi$ and a coordinate system $(x_i, u_i)_{i=1,2,3}$ on $T^*\mathbb{R}^6$ such that

$$g_\omega = \sum_{i,j} \frac{\partial^2 \Phi}{\partial x_i \partial \xi_j} dx_i \cdot d\xi_j$$

and

$$\det\left(\frac{\partial^2 \Phi}{\partial x_j \partial \xi_j}\right) = f(x)g(\xi).$$

## 4.2 Reduction principle in six dimensions

The Marsden-Weinstein reduction process is a classical tool in symplectic geometry to reduce spaces with symmetries (see for example Marsden 1992). It is explained in Banos (2015) how the theory of Monge-Ampère operators gives a geometric formalism in which the symplectic reduction process of Marsden-Weinstein fits naturally to reduce equations with symmetries, and we recall the basic ideas here.

Let us consider a Hamiltonian action $\lambda : G \times M \to M$ of a 1-dimensional Lie group $G$ ($G = \mathbb{R}$, $\mathbb{R}^*$ or $S^1$) on our symplectic manifold $(M = T^*\mathbb{R}^3, \Omega)$. We denote by $X$ the infinitesimal generator of this action and $\mu : M \to \mathbb{R}$ is the moment map of this action:

$$\iota_X \Omega = -d\mu.$$

For a regular value $c$, we know from the Mardsen-Weinstein reduction theorem that the 4-dimensional reduced space $M_c = \mu^{-1}(c)/G$ admits a natural symplectic form $\Omega_c$ such that $\pi^*\Omega_c$ is the restriction of $\Omega$ on $\mu^{-1}(c)$, where $\pi : \mu^{-1}(c) \to M_c$ is the natural projection.

Assume moreover that the action $\lambda$ preserves also the 3-form $\omega$:

$$\forall g \in G, \ \ \lambda_g^*(\omega) = \omega.$$

Then there exists a 2-form $\omega_c$ on $M_c$ such that

$$\pi^*(\omega_c) = \iota_X \omega.$$

We then obtain a Monge-Ampère structure $(\Omega_c, \omega_c)$ on the reduced space $M_c$, and there is a one-to-one correspondence between regular solutions $L_c$ on $M_c$ (that is $\Omega_c|_{L_c} = 0$ and $\omega_c|_{L_c} = 0$) and $G$-invariant generalized solutions $L = \pi^{-1}(L_c)$ on $M$.

**Example** We consider as a trivial example the $3D$-Laplace equation

$$\phi_{x_1x_1} + \phi_{x_2x_2} + \phi_{x_3x_3} = 0.$$

The corresponding Monge-Ampère structure is $(\Omega, \omega)$, which in local coordinates $x_i, \xi_i$ is

$$\Omega = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2 + dx_3 \wedge d\xi_3$$

and

$$\omega = d\xi_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge d\xi_2 \wedge dx_3 + dx_1 \wedge dx_2 \wedge d\xi_3.$$

Let $G = \mathbb{R}$ act on $T^*\mathbb{R}^3$ by translation on the third coordinate:

$$\lambda_\tau(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3 + \tau, \xi_1, \xi_2, \xi_3).$$

This action is trivially Hamiltonian with moment map $\mu(x, \xi) = -\xi_3$ and infinitesimal generator $X = \lambda_* \left(\frac{d}{d\tau}\right) = \frac{\partial}{\partial x_3}$.

For $c \in \mathbb{R}$, the reduced space $M_c$ is canonically $\mathbb{R}^4$ and the reduced Monge-Ampère structure is $(\Omega_c, \omega_c)$ with

$$\Omega_c = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2$$

and

$$\omega_c = \iota_{\frac{\partial}{\partial x_3}} \omega = d\xi_1 \wedge dx_2 + dx_1 \wedge d\xi_2.$$

Then we obtain the Laplace equation in two dimensions as the reduced equation.

# 5 The geometry of Burgers'-type vortices in three dimensions

## 5.1 3D-2D reduction in fluid dynamics: introduction

It is well known in fluid dynamics that certain solutions of the incompressible Euler (and Navier–Stokes) equations in three dimensions can be mapped to solutions of the incompressible equations in two dimensions using the so-called Lundgren transformation. The class of solutions for which this mapping is possible is characterised by the linearity of the third component of velocity in the third spatial coordinate, with the other two components of the velocity being independent of the third spatial coordinate. The canonical Burgers' vortex is a member of this family of solutions, and in this section we show how the formulation described in Section 2.2 can be applied to illuminate the geometry of these solutions.

Joyce (2005) used the reduction principle to construct examples of $U(1)$-invariant special lagrangian manifolds in $\mathbb{C}^3$ after reducing the 6-dimensional "complex" Calabi-Yau structure. The 4-dimensional geometry can be seen as a symplectic reduction of a 6-dimensional "real Calabi-Yau" geometry, which we describe now.

In this section we are concerned with solutions of the incompressible Euler equations of Burgers'-type. That is, we consider flows of the form

$$u_i = (-\gamma(t)x_1/2 - \psi_{x_2}, -\gamma(t)x_2/2 + \psi_{x_1}, \gamma(t)x_3)^T, \tag{12}$$

which satisfy incompressibility $u_{ii} = 0$, and $\psi$, a stream function, is independent of $x_3$. Here $\gamma(t)$ is the strain, which is a function of time alone. We shall consider the analogue of (4) for this class of flows in three dimensions and, to this end, it is convenient to introduce a new stream function

$\Psi(x_1, x_2, x_3, t)$ as follows

$$\Psi = \psi(x_1, x_2, t) - \frac{3}{8}\gamma(t)^2 x_3^2. \tag{13}$$

### 5.2 Extended real Calabi-Yau geometry

We now extend the 4-dimensional geometry associated with (4) to a 6-dimensional geometry.

Using (12) and (13), (3) becomes

$$\Psi_{x_1x_1}\Psi_{x_2x_2} - \Psi_{x_1x_2}^2 + \Psi_{x_3x_3} = \frac{\Delta p}{2} \tag{14}$$

where the Laplace operator on the right hand side is in three dimensions, but note (from (13)) that $\Delta p$ is independent of $x_3$ (cf. Ohkitani and Gibbon 2000).

Denote by $(\Omega, \varpi)$ the corresponding Monge-Ampère structure on $\mathbb{T}^*\mathbb{R}^3$, with $\Omega$ the canonical symplectic form

$$\Omega = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2 + dx_3 \wedge d\xi_3,$$

where $\xi_i = \nabla_i \Psi$ and, following Roulstone *et al.* (2009), $\varpi$ is the effective 3-form

$$\varpi = d\xi_1 \wedge d\xi_2 \wedge dx_3 - a dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_2 \wedge d\xi_3, \tag{15}$$

with $a(x_1, x_2) = \Delta p/2$.

The corresponding "real Calabi-Yau" structure on $T^*\mathbb{R}^6$ is

$$(g_\varpi, K_\varpi, \Omega, \varpi + \hat{\varpi}, \varpi - \hat{\varpi})$$

with

i) $g_\varpi = 2a dx_3 \otimes dx_3 + dx_1 \otimes d\xi_1 + dx_2 \otimes d\xi_2 - dx_3 \otimes d\xi_3 \qquad (\varepsilon(g_\varpi) = (3,3))$

ii) $K_\varpi = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2a & 0 & 0 & -1 \end{pmatrix}$ and thus $\quad K_\varpi^2 = 1$

iii) $\varpi + \hat{\varpi} = 2 d\xi_1 \wedge d\xi_2 \wedge dx_3, \quad \varpi - \hat{\varpi} = 2 dx_1 \wedge dx_2 \wedge (d\xi_3 - a dx_3),$

and the Hitchin pfaffian is $\lambda(\varpi) = 1$.

**PROPOSITION 4.** *1. This "real Calabi-Yau" structure is integrable: locally there exists a potential $F$ and coordinates $(x, \xi)$ in which*

$$g_\varpi = \sum_{j=1}^{3}\sum_{k=1}^{3} \frac{\partial^2 F}{\partial x_j \partial \xi_k} dx_j \cdot d\xi_k$$

*and*

$$\Omega = \sum_{j=1}^{3}\sum_{k=1}^{3} \frac{\partial^2 F}{\partial x_j \partial \xi_k} dx_j \wedge d\xi_k$$

*2. The pseudo-metric $g_\varpi$ is Ricci-flat.*

*3. It is flat if and only if $\Delta p = \alpha_1 x_1 + \alpha_2 x_2 + \beta$.*

*Proof.* Since $d\varpi = d\hat{\varpi} = 0$, we obtain immediately integrability. Direct computations show that the pseudo metric $g_\varpi$ has a curvature tensor which is null if and only if $a(x_1, x_2) = \Delta p/2$ is "affine", whereas Ricci-curvature always vanishes.

□

**REMARK 2.** *From Proposition 3, we deduce that equation* (14) *is equivalent to* $\operatorname{hess}\psi = 1$ *if and only if $\Delta p = \alpha_1 x_1 + \alpha_2 x_2 + \beta$, where $\alpha_1, \alpha_2$ and $\beta$ are constants.*

### 5.3 The symplectic reduction

Let $G = \mathbb{R}$ acting on $T^*\mathbb{R}^3$ by translation on the third coordinate:

$$\lambda_\tau(x_1, x_2, x_3, \xi_1, \xi_2, \xi_3) = (x_1, x_2, x_3 + \tau, \xi_1, \xi_2, \xi_3)$$

This action is trivially hamiltonian with moment map $\mu(x, \xi) = -\xi_3$ and infinitesimal generator $X = \lambda_* \left(\frac{d}{d\tau}\right) = \frac{\partial}{\partial x_3}$.

For $c \in \mathbb{R}$, the reduced space $M_c$ is the quotient space

$$\begin{aligned} M_c &= \mu^{-1}(c)/\mathbb{R} \\ &= \left\{(x_1, x_2, x_3, \xi_1, \xi_2, -c),\ (x_1, x_2, x_3, \xi_1, \xi_2) \in \mathbb{R}^5\right\}/x_3 \sim x_3 + \tau \\ &\cong \left\{(x_1, x_2, \xi_1, \xi_2, -c),\ (x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4\right\} \end{aligned}$$

The vector $Y = K_\varpi X = \dfrac{\partial}{\partial x_3} + 2a\dfrac{\partial}{\partial \xi_3}$ satisfies $\Omega(X,Y) = 2a$ with $a(x_1,x_2) = \Delta p/2$. Therefore, when $\Delta p \neq 0$, one can define a 4-dimensional distribution $D$ which is $\Omega$-orthogonal to $\mathbb{R}X \oplus \mathbb{R}Y$:

$$T_m\mathbb{R}^6 = D_m \oplus_\perp (\mathbb{R}X_m \oplus \mathbb{R}Y_m)$$

with $m = (x,\xi) \in \mathbb{R}^6 = T^*\mathbb{R}^3$, and $\Omega$ and $\varpi$ decompose as follows:

$$\begin{cases} \Omega = \Omega_c - \dfrac{1}{2a}\iota_X\Omega \wedge \iota_Y\Omega \\ \\ \varpi = \varpi_1 \wedge \iota_X\Omega + \varpi_2 \wedge \iota_Y\Omega. \end{cases}$$

Here, $\Omega_c, \varpi_1$ and $\varpi_2$ in $\Omega^2(D)$ are defined by

$$\begin{cases} \Omega_c = dx_1 \wedge d\xi_1 + dx_2 \wedge d\xi_2 \\ \\ \varpi_1 = -\dfrac{1}{2a}(d\xi_1 \wedge d\xi_2 - a dx_1 \wedge dx_2), \\ \\ \varpi_2 = \dfrac{1}{2a}(d\xi_1 \wedge d\xi_2 + a dx_1 \wedge dx_2). \end{cases}$$

Using the natural isomorphism $T_{[m]}M_c \cong D_m$, we get a triple $(\Omega_c, \varpi_1, \varpi_2)$ on $M_c$ which is, up to a re-normalization, the hypersymplectic triple on $T^*\mathbb{R}^4$ associated with the $2D$-Euler equation (cf. §3.2).

# 6 Reduction of incompressible 3D-Euler equations

## 6.1 Introduction: Monge-Ampère geometry of a non-Monge-Ampère equation

We now look at more general solutions to the incompressible equations in three dimensions. The Burgers' vortex is a special solution in that we may define a stream function (13) and use the theory described in §5.2 and Banos (2002). However, for general incompressible flows in three dimensions a single scalar stream function is not available and we have to work with a vector potential. Therefore we should stress that there is no underlying MA equation. We now proceed to show how a geometry, very similar to the one described earlier, can be recovered for three-dimensional flows with symmetry. Our results extend those of Roulstone *et al.* (2009b).

We introduce the following 3-forms $\omega$ and $\theta$ on the phase space $T^*\mathbb{R}^3$, with local coordinates $(x_i, u_i)$ (so here we treat the $u_i$ as a velocity, and these coordinates replace the $\xi_i$; more rigorously, we have identified $T\mathbb{R}^3$ and $T^*\mathbb{R}^3$ using the euclidean metric):

$$\omega = a\, dx_1 \wedge dx_2 \wedge dx_3 - du_1 \wedge du_2 \wedge dx_3 - du_1 \wedge dx_2 \wedge du_3 - dx_1 \wedge du_2 \wedge du_3 \quad (16)$$

with

$$a = \frac{\Delta p}{2}$$

and

$$\theta = du_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge du_2 \wedge dx_3 + dx_1 \wedge dx_2 \wedge du_3. \quad (17)$$

Considering $u = (u_1, u_2, u_3)$ as a 1-form $u = u_1 dx_1 + u_2 dx_2 + u_3 dx_3$ on $\mathbb{R}^3$, it is straightforward to check that $u$ is a solution of $-\Delta p = u_{ij} u_{ji}, \ \ u_{ii} = 0$ if and only if the graph $L_u = \{(x, u)\} \subset T^*\mathbb{R}^3$ is bilagrangian with respect to $\omega$ and $\theta$; that is

$$\omega|_{L_u} = 0 \qquad \text{and} \qquad \theta|_{L_u} = 0. \quad (18)$$

We observe that a solution here is not necessarily lagrangian with respect to the underlying symplectic form $\Omega$, which is compatible with the fact that our equation is no longer of Monge-Ampère form.

## 6.2 Hitchin tensors and Lychagin-Roubtsov metrics

Note that

$$\omega \wedge \theta = 3\, \text{vol} \quad \text{with} \quad \text{vol} = dx_1 \wedge dx_2 \wedge dx_3 \wedge du_1 \wedge du_2 \wedge du_3.$$

We can therefore define the following analogues of Hitchin tensors $K_\omega$ and $K_\theta$:

$$< \alpha, K_\omega(X) > = \frac{\iota_X(\omega) \wedge \omega \wedge \alpha}{\text{vol}} \quad , \quad < \alpha, K_\theta(X) > = \frac{\iota_X(\theta) \wedge \theta \wedge \alpha}{\text{vol}}$$

for any 1-form $\alpha$ and any vector field $X$.

In the $3 \times 3$ decomposition $(x, u)$, these tensors are

$$K_\omega = 2 \begin{pmatrix} 0 & -\text{Id} \\ a\,\text{Id} & 0 \end{pmatrix} \ , \quad K_\theta = 2 \begin{pmatrix} 0 & 0 \\ \text{Id} & 0 \end{pmatrix}. \quad (19)$$

We have the relations

$$\begin{aligned} K_\omega^2 &= -4a\,\mathrm{Id} \\ K_\theta^2 &= 0 \\ K_\omega K_\theta + K_\theta K_\omega &= -4\mathrm{Id} \\ [K_\omega, K_\theta] &= 4\begin{pmatrix} -\mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \end{aligned} \tag{20}$$

Moreover, using the symplectic form $\Omega = dx_1\wedge du_1 + dx_2\wedge du_2 + dx_3\wedge du_3$, one can also define the analogues of Lychagin-Roubtsov metrics $g_\omega = \Omega(K_\omega\cdot,\cdot)$ and $g_\theta = \Omega(K_\theta\cdot,\cdot)$:

$$g_\omega = \begin{pmatrix} 2a\,\mathrm{Id} & 0 \\ 0 & 2\mathrm{Id} \end{pmatrix} \quad , \quad g_\theta = \begin{pmatrix} 2\mathrm{Id} & 0 \\ 0 & 0 \end{pmatrix}. \tag{21}$$

## 6.3 The symplectic reduction

Let us consider an hamiltonian action $\lambda$ of a 1-dimensional Lie group $G$ on our 6-dimensional phase space with moment map $\mu$. Assume moreover that

1. the action $\lambda$ preserves also $\omega$ and $\theta$:

$$\forall g \in G, \lambda_g^*(\omega) = \omega \quad \text{and} \quad \lambda_g^*(\theta) = \theta$$

2. the infinitesimal generator $X$ satisfies $g_\omega(X,X) \neq 0$ and $g_\theta(X,X) \neq 0$

NB. We impose that the action preserves the symplectic form $\Omega$ to obtain a reduced space, but as we already mentioned, this symplectic form is additional data in this context.

As we have seen, there exists a pair $(\omega_c, \theta_c)$ of 2-forms on $M_c$ such that

$$\pi^*(\omega_c) = \iota_X\omega \quad , \quad \pi^*(\theta_c) = \iota_X\theta \tag{22}$$

**REMARK 3.** *if $L_c$ is a 2-dimensional bilagrangian submanifold of $M_c$ (that is $\omega_c|_{L_c} = 0$ and $\theta_c|_{L_c} = 0$) then $L = \pi^{-1}(L_c)$ is "bilagrangian" with respect to $\omega$ and $\theta$.*

*Hence, there is a correspondence between bilagrangian surfaces in the reduced space $M_c$ and $G$-invariant solutions of* (18).

**REMARK 4.** *One can check that the largest group preserving $\Omega$, $\omega$ and $\theta$ is $SO(3,\mathbb{R})$ acting on $M$ by*

$$A\cdot(x,u) = (A\cdot x, A^{-1}\cdot u).$$

### 6.4 Action by translation

Let $\mathbb{R}$ acting on $T^*\mathbb{R}^3$ by translation on $(x_3, u_3)$:

$$\lambda_a(x_1, x_2, x_3, u_1, u_2, u_3) = (x_1, x_2, x_3 + \tau, u_1, u_2, u_3 + \gamma\tau)\ , \quad \gamma \in \mathbb{R}$$

The infinitesimal generator is

$$X = \lambda_* \left(\frac{d}{d\tau}\right) = \frac{\partial}{\partial x_3} + \gamma \frac{\partial}{\partial u_3}$$

and the moment map is

$$\mu(x, u) = \gamma\, x_3 - u_3.$$

We observe that $\mu =$ constant yields the linearity of $u_3$ in $x_3$, as defined in (12).

For $c \in \mathbb{R}$, the reduced space $M_c$ is trivially $T^*\mathbb{R}^2$:

$$\begin{aligned} M_c &= \{(x_1, x_2, x_3, u_1, u_2, \gamma x_3 - c\} / x_3 \sim x_3 + t \\ &= \{(x_1, x_2, 0, u_1, u_2, -c)\} \end{aligned}$$

and the pair $(\omega_c, \theta_c) = (\iota_X \omega, \iota_X \theta)$ is

$$\begin{cases} \omega_c = a\, dx_1 \wedge dx_2 - du_1 \wedge du_2 - \gamma du_1 \wedge dx_2 - \gamma dx_1 \wedge du_2 \\ \theta_c = du_1 \wedge dx_2 + dx_1 \wedge du_2 + \gamma dx_1 \wedge dx_2 \end{cases}$$

Considering the following change of variables

$$\begin{aligned} X_1 &= x_1\ , \qquad U_1 = \frac{\gamma}{2} x_2 + u_2, \\ X_2 &= x_2\ , \qquad U_2 = -\frac{\gamma}{2} x_1 - u_1, \end{aligned}$$

we obtain

$$\theta_c = dX_1 \wedge dU_1 + dX_2 \wedge dU_2 \tag{23}$$

and

$$\omega_c = \omega_0 - \frac{\gamma}{2} \theta_c \tag{24}$$

with

$$\omega_0 = \left(a + \frac{3}{4}\gamma^2\right) dX_1 \wedge dX_2 - dU_1 \wedge dU_2. \tag{25}$$

Hence we summarize our result as follows:

**PROPOSITION 5.** *If $\psi(x_1, x_2, t)$ is solution of the Monge-Ampère equation in two dimensions*

$$\psi_{x_1x_1}\psi_{x_2x_2} - \psi^2_{x_1x_2} = \frac{\Delta p(x_1, x_2, t)}{2} + \frac{3}{4}\gamma^2(t). \tag{26}$$

*then the velocity $u(x_1, x_2, x_3, t)$ defined by*

$$\begin{aligned} u_1 &= -\frac{\gamma(t)}{2}x_1 - \psi_{x_2}, \\ u_2 &= -\frac{\gamma(t)}{2}x_2 + \psi_{x_1}, \\ u_3 &= \gamma(t)x_3 - c(t), \end{aligned}$$

*is solution of $-\Delta p = u_{ij}u_{ji}$ and $u_{ii} = 0$ in three dimensions.*

*Proof.* Time $t$ is a parameter here, hence $c = c(t)$ and $\gamma = \gamma(t)$ are constant. If $\psi(x_1, x_2, t)$ is solution of (26) then $\omega_0|_\psi = \theta_c|_{L_\psi} = 0$, with

$$L_\psi = \{(X_1, X_2, \psi_{X_1}, \psi_{X_2})\}.$$

From (24), we deduce that $\omega_c|_{L\psi} = \theta_c|_{L_\psi} = 0$, and therefore $\omega|_L = \theta|_L = 0$ with

$$L = \pi^{-1}(L_c) = \left\{(x_1, x_2, x_3, -\frac{\gamma}{2}x_1 - \psi_{x_2}, -\frac{\gamma}{2}x_1 + \psi_{x_1}, \gamma x_3 - c)\right\}.$$

□

# 7 Summary

The Poisson equation for the pressure in the incompressible Euler and Navier–Stokes equations can be written in the form

$$\Delta p = \frac{\zeta^2}{2} - \mathrm{Tr} S^2, \tag{27}$$

where $\zeta$ is the vorticity and $S$ is the rate-of-strain matrix. Equation (27) illuminates the relationship between the sign of the Laplacian of the pressure and the balance between vorticity and strain in the flow. Gibbon (2008) has conjectured that much might be learned from studying the geometric properties of this equation, which *locally* holds the key to the formation of vortical structures — ubiquitous features of both turbulent flows and large-scale atmosphere/ocean dynamics.

In this paper we have studied (27) through Monge-Ampère structures and their associated geometries. Burgers' vortex, which has been described as the *sinew of turbulence*, is shown to arise naturally from a symmetry reduction of a Monge-Ampère structure (Section 5). This Monge-Ampère structure is itself derived from a symmetry reduction of the equations for incompressible flow in three dimensions (§6.3), and mapping to a solution of 2d incompressible flow has been presented (Proposition 5). This is surely associated with the well-known Lundgren transformation.

In contrast to earlier works by the same authors, we have shown explicity how the geometries associated with both signs of the Laplacian of the pressure can be described by almost-complex and almost-product structures. This points to a possible role for generalised geometry (Banos 2007), and this is a topic for future research.

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